A. TRIGONOMETRY

Introduction

The word ‘trigonometry’ is derived from two Greek words ‘trigonon’ and ‘metron’. Trigonon means a triangle and metron means a measure. Hence, trigonometry means measurement of triangles, i.e. study of triangles, measurement of their sides, angles and different relations which exist between triangles relates. Initially, this concept was developed to solve geometric problems involving triangles. In earlier days, these ideas were used by sea captains for navigations, surveyor to map the lands, architects/ engineers to construct the buildings, dams, etc and others. But now a days, it’s application has extended to many areas like satellite navigations, seismology, measurement of height of a building or mountain, in video games, construction and architecture, flight engineering, cartography (creating maps),in oceanography to measure height of the tides and many other areas also.

The following three different systems of units are used in the measurement of trigonometrical angles

Measurement of an angle:

There are three systems of measurement of an angle.

(i) Sexagesimal system
(ii) Centesimal system
(iii) Circular system

(i) Sexagesimal system

(i) $1\text{right angle} = 90 \text{ degrees}(90°)$
(ii) $1° = 60 \text{ sexagesimal minutes or}(60')$
(iii) $1 \text{ minute or } 1' = 60 \text{ sexagesimal seconds or } 60''$

(ii) Centesimal system

(i) $1 \text{ right angle} = 100\text{grades or } 100^{g}$
(ii) $1^{g} = 100 \text{ centesimal minutes}$
(iii) $1\text{right angle} = 90° = 100^{g}$

(iii) Circular system

The unit of measurement of angles in this system is a radian. A radian is the angle subtended at the centre of a circle by an arc whose length is equal to the radius of that circle and is denoted by 1^{c}.

$$\frac{\text{circumference}}{\text{diameter}} = \pi,$$

π is a Greek letter, pronounced by “pi” .Two right angles =$180° = 200^{g} = \pi^{c}$

$$1 \text{ radian} = \frac{2}{\pi} \text{ right angle.}$$

NOTE :(i) Angle subtended by an arc length l is $\theta = \frac{l}{r}$

(ii) The angle subtended at the centre of a circle in radians is 2π radians

Trigonometric Ratios

Let XOX' and YOY' be two axes of co-ordinates i.e. x-axis and y-axis respectively. These two axes intersect perpendicularly at the point 'O', named as Origin.

In x-axis, OX and OX' are known as positive and negative X-axis respectively. Similarly, In y-axis, OY and OY' are known as positive and negative Y-axis respectively. Now, both the axes divide the XY-plane into four equal parts called'quadrants'.

(i) XOY is called 1st quadrant.
(ii) X'OY is called 2nd quadrant..
(iii) X'OY' is called 3rd quadrant.
(iv) XOY' is called 4th quadrant.

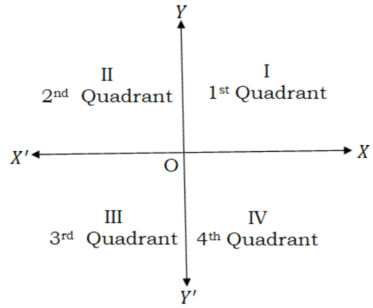


Fig. 2.1

Let us take a point P(x, y). Draw $PM \perp OX$
In Fig 2.2, join OP.
S0 that, $OM = x$, $PM = y$
OPM is a right angle triangle.
If θ is an angle measure such that $0 < \theta < \pi/2$.
Let $\angle POM = \theta$.
So, the side OP opposite of the right angle $\angle PMO$ is known as hypotenuse (h).
The side OM related to right angle and given angle(θ) is base(b) and the side PM is known as the perpendicular(p).

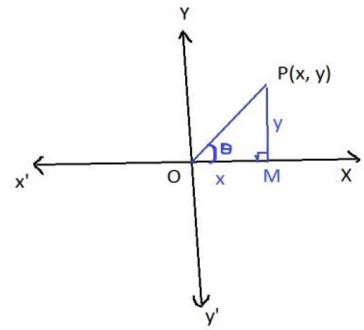


Fig 2.2

Now, in the right angle triangle ∆OPM, (Fig 2.2)
The ratio of its sides (with proper sign) are defined as trigonometrical ratios (T-Ratios).
There are six trigonometric ratios such as sine, cosine, tangent, cotangent, secant, cosecant for θ, abbreviated as sinθ, cosθ, tanθ, cot θ, secθ, cosecθ have been defined as follows.

1. The ratio of the perpendicular to the hypotenuse, is called "sine of the angle θ" and it is written as $\sin\theta$.
   i.e. $\sin\theta = \frac{Perpendicular}{\text{hypotenuse}} = \frac{PM}{OP}$
2. The ratio of the base to the hypotenuse, is called "cosine of the angle θ" and it is written as $\cos\theta$.
   i.e. $\cos\theta = \frac{Base}{\text{Hypotenuse}} = \frac{PM}{OP}$
3. The ratio of the perpendicular to the base, is called "tangent of the angle θ" and it is written as $\tan\theta$.
   i.e. $\tan\theta = \frac{Perpendicular}{Base} = \frac{PM}{OM}$
4. The ratio of the hypotenuse to perpendicular, is called "cosecant of the angle θ" and it is written as $\text{cosec}\,\theta$.
   i.e. $\text{cosec}\,\theta = \frac{\text{hypotenuse}}{Perpendicular} = \frac{OP}{PM}$

5. The ratio of the hypotenuse to base, is called “secant of the angle θ” and it is written as $\sec\theta$.

   i.e. $\sec\theta = \frac{\text{Hypotenuse}}{Base} = \frac{OP}{OM}$.

6. The ratio of the base to perpendicular, is called “cotangent of the angle θ” and it is written as $\cot\theta$.

   i.e. $\cot\theta = \frac{Base}{Perpendicular} = \frac{OM}{PM}$.

Notes :

i**.** All the above six ratios are called trigonometrical ratios (T-Ratios).

ii. $cosec\theta = \frac{1}{\sin\theta}$. Or, $sin\,\theta$ and $cosec\theta$ are reciprocal ratios.

$sec\theta = \frac{1}{cos\theta}$, Or, $cos\,\theta$ and $sec\theta$ are reciprocal ratios.

$\cot\theta = \frac{1}{tan\theta}$, Or, $tan\theta$ and $\cot\theta$ are reciprocal ratios.

iii. For angle measure $\pi/2$, we define

$$sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0, cot\frac{\pi}{2} = 0, cosec\frac{\pi}{2} = 1,$$

$tan\frac{\pi}{2}$ and $sec\frac{\pi}{2}$ are not defined.

$sin\frac{\pi}{2}, \cos\frac{\pi}{2}, cot\frac{\pi}{2}, cosec\frac{\pi}{2}$ are not defined as ratios of sides. So instead of using trigonometric ratios, we use a more general form for them, called as trigonometric functions, in due course. For the same reason we do not use the term trigonometric ratio for $sin0^{\circ}$, $\cos 0^{\circ}$, $tan0^{\circ}$and $sec0^{\circ}$. However we define

$$sin0^{\circ} = 0, \cos 0^{\circ} = 1, tan0^{\circ} = 0 \text{ and } sec0^{\circ} = 1.$$

Trigonometric functions:

The six trigonometric functions are given by the following.

(i) sine: $R \to [-1, 1]$,

(ii) $cosine: R \to [-1,1]$,

(iii) $tangent: R - \{(2n+1)\frac{\pi}{2} : n \in Z \to R$

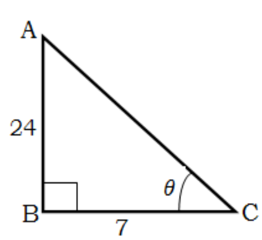
(iv) $cotangent: R - \{n\pi : n \in Z\} \to R$

(v) $secant: R - \left\{(2n+1)\frac{\pi}{2} : n \in Z\right\} \to R - (-1,1)$

(vi) $cosecant: R - \{n\pi : n \in Z\} \to R - (-1, 1)$

For θ = (2n+1)$\frac{\pi}{2}$$: n \in Z$,tan$\theta$,s$ec\,\theta$ does not exist. Similarly for $\theta = n\pi : n \in Z$ cot θ, cosec θ does not exist ,for this reason those terms are excluded from the respective functions.

Example-1: In ∆ABC, right angle is at B and AB=24 cm, BC=7cm.



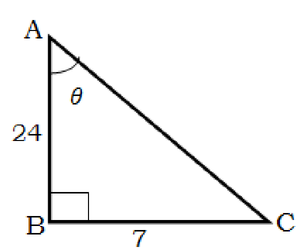


Fig. 1

Fig. 2.5 Fig 2.6

(i) In Fig-1, $\angle B = 90^{\circ}$ and $\angle C = \theta$ is the given angle.

So, Base = $b = BC = 7cm$, Hypotenuse = $h = AC$

and Perpendicular= $p = AB = 24cm$.

By using Pythagoras theorem, $p^2 + b^2 = h^2$

$\therefore\ h^2 = 24^2 + 7^2 = 625$ and $h = 25cm$

Therefore, $\sin A = \frac{p}{h} = \frac{24}{25}$, $\cos A = \frac{b}{h} = \frac{7}{25}$ and $\tan A = \frac{p}{b} = \frac{24}{7}$

(ii) But, In Fig-2, $\angle B = 90^{\circ}$ and $\angle A = \theta$ is the given angle.

So, Base = $b = AB = 7cm$, Hypotenuse = $h = AC$

and Perpendicular= $p = BC = 7cm$.

By using Pythagoras theorem, $p^2 + b^2 = h^2$

$\therefore\ h^2 = 24^2 + 7^2 = 625$ and $h = 25cm$

(iii) Therefore, $\sin A = \frac{p}{h} = \frac{7}{25}$, $\cos A = \frac{b}{h} = \frac{24}{25}$ and $\tan A = \frac{p}{b} = \frac{7}{24}$.

Example-2: Let $cot\theta = \frac{7}{8}$,

So, $cot\theta = \frac{b}{p} = \frac{7}{8} = k$, where k is a proportionality constant.

$\therefore\ b = 7k$ and $p = 8k$

By using Pythagoras theorem: $p^2 + b^2 = h^2$

$$\Rightarrow (8k)^2 + (7k)^2 = h^2$$

$$\Rightarrow h^2 = 113k^2 \ \ or\ \ h = \sqrt{113}k$$

Hence, $sec\theta = \frac{h}{b} = \frac{\sqrt{113}\ k}{7k} = \frac{\sqrt{113}}{7}$ and $cosec\ \theta = \frac{h}{p} = \frac{\sqrt{113}}{8}$.

Trigonometry Identity

(i) $\boldsymbol{sin^2\theta + cos^2\theta = 1}$

(ii) $\boldsymbol{sec^2\theta - tan^2\theta = 1}$

(iii) $\boldsymbol{cosec^2\theta - cot^2\theta = 1}$

Proof:

(i) LHS: $sin^2\theta + cos^2\theta = (sin\theta)^2 + (cos\theta)^2 = \left(\frac{p}{h}\right)^2 + \left(\frac{b}{h}\right)^2$

$= \frac{p^2}{h^2} + \frac{b^2}{h^2} = \frac{p^2+b^2}{h^2} = \frac{h^2}{h^2} = 1$ =RHS

[Note: By Pythagoras Theorem,

In a right angled triangle, $p^2 + b^2 = h^2$.]

(ii) LHS: $sec^2\theta - tan^2\theta = (sec\theta)^2 - (tan\theta)^2 = \left(\frac{h}{b}\right)^2 - \left(\frac{p}{b}\right)^2$

$= \frac{h^2}{b^2} - \frac{p^2}{b^2} = \frac{h^2-p^2}{b^2} = \frac{b^2}{b^2} = 1$=RHS

(iii) LHS: $cosec^2\theta - cot^2\theta = (cosec\theta)^2 - (cot\theta)^2 = \left(\frac{h}{p}\right)^2 - \left(\frac{b}{p}\right)^2$

$= \frac{h^2}{p^2} - \frac{b^2}{p^2} = \frac{1}{p^2}(h^2 - b^2) = \frac{1}{p^2}(p^2) = 1$=RHS

Note: All the above relations/identities hold good for any value of θ. i.e. these identities are independent of the angle(θ). i.e. whatever may be the angle, the relations are true.

Example : $sin^2x + cos^2x = 1$, $cosec^2\alpha - cot^2\alpha = 1$, $sec^2\beta - tan^2\beta = 1$, *etc.*

Some Solved Problems

Q.1: Prove $\frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta} = 2\ cosec\theta$

Proof:

LHS =$\frac{sin\theta}{1+cos\theta} + \frac{1+cos\theta}{sin\theta}$

$= \frac{sin\theta.sin\theta + (1+cos\theta)(1+cos\theta)}{(1+cos\theta)sin\theta}$

$= \frac{sin^2\theta + (1+cos\theta)^2}{(1+cos\theta)sin\theta}$

$= \frac{sin^2\theta + 1+cos^2\theta+2cos\theta}{(1+cos\theta)sin\theta}$ $[\therefore\ sin^2\theta + cos^2\theta = 1$

$= \frac{1+1+2cos\theta}{(1+cos\theta)sin\theta}$

$= \frac{2}{sin\theta} = 2cosec\theta = RHS$

Q-2: Prove $\frac{1}{1-sinx} + \frac{1}{1+sinx} = 2sec^2x$

Proof:

LHS= $\frac{1}{1-sinx} + \frac{1}{1+sinx}$

$= \frac{1(1+sinx)+1(1-sinx)}{(1-sinx)(1+sinx)}$

$= \frac{1+sinx+1-sinx}{1-sin^2x}$ $[\therefore\ (a-b)(a+b) = a^2 - b^2]$

$= \frac{2}{cos^2x}$ $[\therefore\ 1 - sin^2x = cos^2x$

$= 2cosec^2x$=RHS

Q-3: Prove $(cosec\ \theta - cot\theta)^2 = \frac{1-cos\theta}{1+cos\theta}$

Proof:

LHS= $(cosec\ \theta - cot\theta)^2$

$= \left(\frac{1}{sin\theta} - \frac{cos\theta}{sin\theta}\right)^2$

$= \left(\frac{1-cos\theta}{sin\theta}\right)^2$

$= \frac{(1-cos\theta)^2}{(1-cos\theta)(1+cos\theta)}$ $[\because sin^2\theta = 1 - cos^2\theta = (1-cos\theta)(1+cos\theta)]$

$= \frac{1-cos\theta}{1+cos\theta}$ =RHS

Q-4: Prove $\frac{cosec\theta}{cosec\theta-1} + \frac{cosec\theta}{cosec\theta+1} = 2sec^2\theta$

Proof :

LHS = $\frac{cosec\theta}{cosec\theta-1} + \frac{cosec\theta}{cosec\theta+1}$

$= \frac{cosec\theta(cosec\theta+1)+cosec\theta(cosec\theta-1)}{(cosec\theta-1)(cosec\theta+1)}$

$= \frac{cosec^2\theta+cosec\theta+cosec^2\theta-cosec\theta}{cosec^2\theta-1}$

$= \frac{2cosec^2\theta}{cot^2\theta} = \frac{2}{sin^2\theta} . \frac{sin^2\theta}{cos^2\theta} = 2sec^2\theta$=RHS

Q-5: Prove $\sqrt{\frac{1+sin\theta}{1-sin\theta}} = sec\theta + tan\theta$

Proof: LHS= $\sqrt{\frac{1+sin\theta}{1-sin\theta}} = \sqrt{\frac{1+sin\theta}{1-sin\theta}\frac{1+sin\theta}{1+sin\theta}} = \sqrt{\frac{(1+sin\theta)^2}{1-sin^2\theta}} = \sqrt{\frac{(1+sin\theta)^2}{cos^2\theta}}$

$= \sqrt{\left(\frac{1+sin\theta}{cos\theta}\right)^2} = \frac{1+sin\theta}{cos\theta} = \frac{1}{cos\theta} + \frac{sin\theta}{cos\theta} = sec\theta + tan\theta$ =RHS

Q-6: Prove $\frac{tanx}{1-cotx} + \frac{cotx}{1-tanx} = secx.cosecx + 1$

Proof:

LHS= $\frac{tanx}{1-cotx} + \frac{cotx}{1-tanx}$

$= \frac{sinx}{cosx}\frac{1}{1-\frac{cosx}{sinx}} + \frac{cosx}{sinx}\frac{1}{1-\frac{sinx}{cosx}}$

$= \frac{sinx}{cosx}\frac{1}{\frac{sinx-cosx}{sinx}} + \frac{cosx}{sinx}\frac{1}{\frac{cosx-sinx}{cosx}}$

$= \frac{sin^2x}{cosx(sinx-cosx)} - \frac{cos^2x}{sinx(sinx-cosx)}$

$= \frac{sin^3x-cos^3x}{sinx\ cosx\ (sinx-cosx)}$ $\begin{bmatrix} \because sin^3x - cos^3x, a^3 - b^3 = (a-b)(a^2+ab+b^2) \\ = (sinx - cosx)(sin^2x + cos^2x + sinx\ cosx) \\ = (sinx - cosx)(1 + sinx.cosx) \end{bmatrix}$

$= \frac{(sinx-cosx)(1+sinx.cosx)}{sinx\ cosx\ (sinx-cosx)}$

$= \frac{1}{sinx\ cosx} + 1 = secx\ cosecx + 1$ =RHS

Q-7: Prove $\frac{sec\theta-tan\theta}{sec\theta+tan\theta} = 1 - 2sec\theta.tan\theta + 2tan^2\theta$

Proof:

LHS: $\frac{sec\theta - tan\theta}{sec\theta + tan\theta}$

$= \frac{sec\theta - tan\theta}{sec\theta + tan\theta} \cdot \frac{sec\theta - tan\theta}{sec\theta - tan\theta}$

$= \frac{(sec\theta - tan\theta)^2}{sec^2\theta - tan^2\theta}$ $[\because sec^2\theta - tan^2\theta = 1]$

$= \frac{(sec\theta - tan\theta)^2}{1}$

$= (1 + tan^2\theta) + tan^2\theta - 2sec\theta . tan\theta$

$= 1 - 2sec\theta . tan\theta + 2tan^2\theta$=RHS

Q-8: Prove that $sin^8\theta - cos^8\theta = (sin^2\theta - cos^2\theta)(1 - 2sin^2\theta . cos^2\theta)$

Proof:

LHS=$sin^8\theta - cos^8\theta$

$= (sin^4\theta)^2 - (cos^4\theta)^2$

$= (sin^4\theta - cos^4\theta)(sin^4\theta + cos^4\theta)$ $[As\ a^2 - b^2 = (a - b)(a + b)]$

$= \{(sin^2\theta)^2 - (cos^2\theta)^2\}\{(sin^2\theta)^2 + (cos^2\theta)^2\}$

$= (sin^2\theta - cos^2\theta)(sin^2\theta + cos^2\theta)\{(sin^2\theta + cos^2\theta)^2 - 2sin^2\theta cos^2\theta\}$

$[\because a^2 + b^2 = (a + b)^2 - 2ab$ and $sin^2\theta + cos^2\theta = 1]$

$= (sin^2\theta - cos^2\theta)(1 - 2sin^2\theta cos^2\theta)$ =RHS

Signs of T-Ratios

Let a revolving line OP, starting from OX trace and $\angle XOP = \theta$. From P, Draw $PM \perp XOX'$.

In the right angled triangle, $OP^2 = OM^2 + PM^2$,

$OP = +\sqrt{OM^2 + PM^2}$

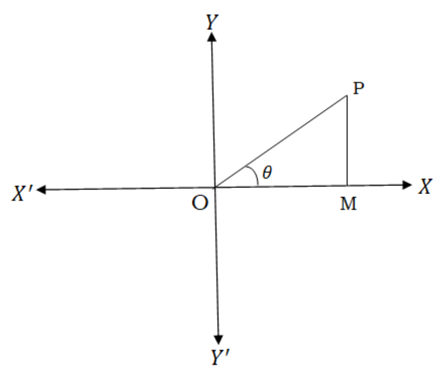


Fig 2.7

Suppose the point 'P' lies in 1st quadrant.

OM=+ve, PM=+ve and OP=+ve

$sin\theta = \frac{PM}{OP} = \frac{+ve}{+ve} = +$ve

$cos\theta = \frac{OM}{OP} = \frac{+ve}{+ve} = +$ve

$tan\theta = \frac{PM}{OM} = \frac{+ve}{+ve} = +$ve

In 1st quadrant, all the T-ratios are having positive signs.

Suppose the point 'P' lies in 2nd quadrant.

OM= $-ve$, PM= $+ve$ and OP= $+ve$

$sin\theta = \frac{PM}{OP} = \frac{+ve}{+ve} = +$ve

$cos\theta = \frac{OM}{OP} = \frac{-ve}{+ve} = -\text{ve}$
$tan\theta = \frac{PM}{OM} = \frac{+ve}{-ve} = -\text{ve}$
In 2nd quadrant, only $sin\theta$ and $cosec\theta$ are positive and all other T-ratios are having negative signs.

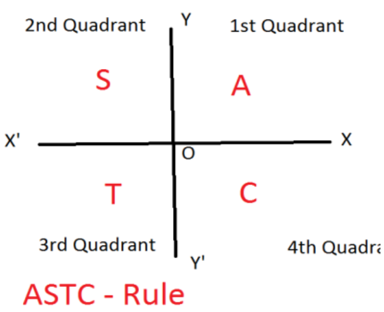
Suppose the point ‘P’ lies in 3rd quadrant.
OM= $-ve$, PM= $-ve$ and $OP = +ve$
$sin\theta = \frac{PM}{OP} = \frac{-ve}{+ve} = -\text{ve}$
$cos\theta = \frac{OM}{OP} = \frac{-ve}{+ve} = -\text{ve}$
$tan\theta = \frac{PM}{OM} = \frac{-ve}{-ve} = +\text{ve}$

In 3rd quadrant, only $tan\theta$ and $cot\theta$ are positive and all other T-ratios are having negative signs.

Suppose the point ‘P’ lies in 4th quadrant.
OM= $+ve$, PM= $-ve$ and $OP = +ve$
$sin\theta = \frac{PM}{OP} = \frac{-ve}{+ve} = -\text{ve}$
$cos\theta = \frac{OM}{OP} = \frac{+ve}{+ve} = +\text{ve}$
$tan\theta = \frac{PM}{OM} = \frac{-ve}{+ve} = -\text{ve}$
In 4th quadrant, only $cos\theta$ and $sec\theta$ are positive and all other T-ratios are having negative signs.

Notes: ASTC-Rule

i. In 1st quadrant, all T-ratios are $+ve$.
ii. In 2nd quadrant, sine is +ve and all others –ve
iii. In 3rd quadrant, tangent is +ve and all others –ve.
iv. In 4th quadrant, cosine is +ve and all others –ve.
v. The sign of any T-Ratio in any quadrant can be recalled by the words ‘all-sin-tan-cos’ or ‘add sugar to coffee’ and this rule is known as ‘ASTC-Rule’. Whatever is written in a particular quadrant, this T-ratio along with its reciprocal are positive and all other ratios are negative.



Trigonometric Ratio of Selected Angles

The values of T-ratios for some selected angles like $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$ are given below..

Angles $\rightarrow$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\text{cosec}\,\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Some Solved Problems

Q-1: Find the value of $\frac{5cos^2 60° + 4sec^2 30^{°} - tan^2 45°}{sin^2 30° + cos^2 30°}$

Sol:

Using the trigonometric values,

$$\frac{5cos^2 60° + 4sec^2 30^{°} - tan^2 45°}{sin^2 30° + cos^2 30°} = \frac{5\left(\frac{1}{2}\right)^2 + 4\left(\frac{2}{\sqrt{3}}\right)^2 - 1^2}{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\frac{5}{4} + \frac{16}{3} - 1}{\frac{1}{4} + \frac{3}{4}} = \frac{\frac{15+64-12}{12}}{\frac{4}{4}} = \frac{67}{12}$$

Q-2: Find the value of $\frac{\cot 45°}{\sqrt{1 - cot^2\, 60°}}$

Sol:

As we know that, $\cot 45^{°} = 1$ and $\cot 60^{°} = \frac{1}{\sqrt{3}}$

Therefore, $\frac{\cot 45°}{\sqrt{1 - cot^2\, 60°}} = \frac{1}{\sqrt{1 - (1/\sqrt{3})^2}} = \frac{1}{\sqrt{2/3}} = \frac{\sqrt{3}}{\sqrt{2}}$

Limits of the values of T-ratios:

1. $-1 \le \sin\theta \le 1,$ and $-1 \le \cos\theta \le 1$
   i.e. the minimum values of both sine and cosine of angles are -1 and maximum values of both sine and cosine of angles are $+1$.
2. $\text{cosec}\,\theta$ and $\sec\theta$ each cannot be numerically less than unity,
   i.e. $-1 \ge cosec\ \theta$ or $cosec\ \theta \ge 1$ and $-1 \ge sec\ \theta$ or $sec\ \theta \ge 1$
3. $\tan\theta$ and $\cot\theta$ can have any numerical value,
   $\tan\theta\ \in R$ and $\cot\theta\ \in R$.

Some Solved Problems

Q-1: Find the maximum and minimum value of $5\ sinx + 12\ cosx$.

Sol:

Let $5 = \text{r cos}\alpha$ and $12 = \text{r sin}\alpha$.

By squaring and adding,

$5^2 + 12^2 = (\text{r cos}\alpha)^2 + (\text{r sin}\alpha)^2$

$\Rightarrow 25 + 144 = r^2(cos^2\alpha + sin^2\alpha)$

$\Rightarrow 169 = r^2.1 = r^2$

$\Rightarrow r = 13$
Now, the given expression $5\ \text{sinx} + 12\ \text{cosx}$ can be reduced to
$\text{r cos}\alpha\ \ \sin x + \text{r sin}\alpha \cos x$
$= r(sinx\ cos\alpha + cosx\ sin\alpha)$
$= r\sin(x+\alpha) = 13\, sin(x+\alpha)$
We know that the minimum and maximum values of $\sin\theta$ are -1 and 1 respectively, i.e. $-1 \le \sin\theta \le 1$
Therefore $\quad -1 \le sin(x+\alpha) \le 1$
Or, $-13 \le 13\, sin(x+\alpha) \le 13$
Hence, the maximum and minimum values of $5\text{sinx} + 12\ \text{cosx}$ are 13 and –13.

Q-2: Find the maximum value of $2 + 3sinx + 4cosx$.
Sol:
Let $3 = rcos\alpha$ and $4 = rsin\alpha$
So, $r = \sqrt{3^2+4^2} = 5$
Now, $3sinx + 4cosx = r\cos\alpha\sin x + \ r\sin\alpha\cos x$
$= r(sinx\ cos\alpha + cosx\ sin\alpha) = 5\sin(x+\alpha)$
We know that, the maximum value of $sin\theta$ =1
∴ Maximum value of $\sin(x+\alpha) = 1$
⇒ Maximum value of $5\sin(x+\alpha) = 5.1 = 5$
⇒ Maximum value of $2 + 5\sin(x+\alpha) = 2 + 5 = 7$

Values of T-ratios of allied angles

1. **T-ratios of $(-\theta)$ in terms of θ, for all values of θ.**
Let OX be the initial line. Let OP be the position of the radius vector after tracing an angle θ in the anticlockwise sense which we take as positive sense. Let OP' be the position of the radius vector after tracing (θ) in the clockwise sense, which we take as negative sense. So $\angle P'OX$ will be taken as $(-\theta)$. Join PP'. Let it meet OX at M.
Here, $\Delta OPM = \Delta P'OM,\ \angle P'OM = -\theta,\ OP' = OP,\ P'M = -PM$
Now $sin(-\theta) = \frac{P'M}{OP'} = \frac{-PM}{OP} = -sin\theta$
$cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = cos\theta$
$tan(-\theta) = \frac{P'M}{OM} = \frac{-PM}{OM} = -tan\theta$
Similarly,
$cot(-\theta) = \frac{OM}{P'M} = \frac{OM}{-PM} = -cot\theta$
$sec(-\theta) = \frac{OP'}{OM} = sec\theta$
$cosec(-\theta) = \frac{OP'}{P'M} = \frac{OP}{-PM} = -cosec\ \theta$

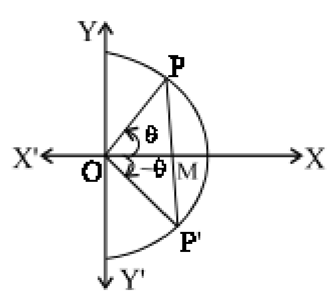


Fig 2.8

Example:
$\sin(-60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$
$\cos(-30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\tan(-45^{\circ}) = -\tan 45^{\circ} = -1$

2. **T-ratios of $(90 - \theta)$ in terms of θ, for all the values of θ**

Let OPM be a right angled triangle with ∠ POM = 90°,

∠OMP =θ, ∠OPM =90°- θ

$sin(90° - \theta) = \frac{OM}{PM} = cos\ \theta$

$cos(90° - \theta) = \frac{OP}{PM} = sin\ \theta$

$tan\ (90° - \theta) = \frac{OM}{OP} = cot\theta$

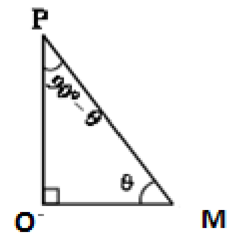


Fig 2.9

Similarly,

$\cot(90 - \theta) = +\tan\theta$

$\sec(90 - \theta) = +\operatorname{cosec}\theta$

$\operatorname{cosec}(90 - \theta) = +\sec\theta$

Here, the angle θ and $90 - \theta$ are called complementary angles and each of the angle is called complement of each other.

Example:

$\sin(90^{\circ} - 30^{\circ}) = +\cos 30^{\circ} = \frac{\sqrt{3}}{2}$

$\cos(90^{\circ} - 60^{\circ}) = +\sin 60^{\circ} = \frac{\sqrt{3}}{2}$

$\tan(90^{\circ} - 45^{\circ}) = +\cot 45^{\circ} = 1$

3. **T-ratios of $(90^{\circ} + \theta)$ in terms of θ, for all the values of θ**

Let $\angle POX = \theta$ and $\angle P'OX = 90° + \theta$. Draw PM and P'M' perpendiculars to the X-axis.

Now ΔPOM ≅Δ P'OM'

∴ P'M' = OM and OM' = –PM

Now $\sin(90° + \theta) = \frac{PM}{OP} = \frac{P'M'}{OP'} = cos\theta$

$\cos(90° + \theta) = \frac{OM'}{OP'} = \frac{-PM}{0P} = -sin\theta$

$\tan(90° + \theta) = \frac{P'M'}{OM'} = \frac{-OM}{PM} = -cot\theta$

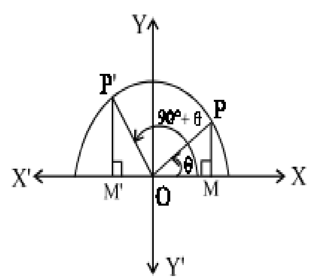


Fig 2.10

Similarly,

$\cot(90° + \theta) = -\tan\theta$

$\sec(90° + \theta) = -\operatorname{cosec}\theta$

$\operatorname{cosec}(90° + \theta) = +\sec\theta$

Examples:

$\sin 120° = \sin(90° + 30°) = +\cos 30° = \frac{\sqrt{3}}{2}$

$\cos 150° = \cos(90° + 60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

$\tan 135° = \tan(90° + 45°) = -\cot 45° = -1$

$\sec 150° = \sec(90° + 60°) = -cosec60° = -\frac{2}{\sqrt{3}}$

Similarly, the values of T-ratios for following allied angles can also be proved.

4. **T-ratios of $(180° - \theta)$ in terms of θ, for all the values of θ**

$\sin(180° - \theta) = +\sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$
$\cot(180° - \theta) = -\cot\theta$
$\sec(180° - \theta) = -\sec\theta$
$\text{cosec}(180° - \theta) = +\text{cosec}\,\theta$

Here, the angle θ and $180° - \theta$ are called supplementary angles and each of the angle is called supplement of each other.

Examples:

$\cos 150° = \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$
$\sec 120° = \sec(180° - 60°) = -\sec 60° = -2$
$\cot 135° = \cot(180° - 45°) = -\cot 45° = -1$

5. **T-ratios of $(180° + \theta)$ in terms of θ, for all the values of θ**

$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = +\tan\theta$
$\cot(180° + \theta) = +\cot\theta$
$\sec(180° + \theta) = -\sec\theta$
$\text{cosec}(180° + \theta) = -\text{cosec}\,\theta$

Examples:

$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$
$\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$
$\tan 240° = \tan(180° + 60°) = +\tan 60° = \sqrt{3}$

6. **T-ratios of $(270° - \theta)$ in terms of θ, for all the values of θ**

$\sin(270° - \theta) = -\cos\theta$
$\cos(270° - \theta) = -\sin\theta$
$\tan(270° - \theta) = +\cot\theta$
$\cot(270° - \theta) = +\tan\theta$
$\sec(270° - \theta) = -\text{cosec}\,\theta$
$\text{cosec}(270° - \theta) = -\sec\theta$

Examples:

$\sin 210° = \sin(270° - 60°) = -\cos 60° = -\frac{1}{2}$
$\cos 240° = \cos(270° - 30°) = -\sin 30° = -\frac{1}{2}$
$\tan 225° = \tan(270° - 45°) = +\cot 45° = +1$

7. **T-ratios of $(270° + \theta)$ in terms of θ, for all the values of θ**

$\sin(270° + \theta) = -\cos\theta$
$\cos(270° + \theta) = +\sin\theta$

$\tan(270° + \theta) = -\cot\theta$
$\cot(270° + \theta) = -\tan\theta$
$\sec(270° + \theta) = +\operatorname{cosec}\theta$
$\operatorname{cosec}(270° + \theta) = -\sec\theta$
Examples:
$\sin 315^{\circ} = \sin(270° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$
$\cos 300^{\circ} = \cos(270° + 30°) = \sin 30^{\circ} = \frac{1}{2}$
$\cot 330^{\circ} = \cot(270° + 60°) = -\tan 60^{\circ} = -\sqrt{3}$

8. **T-ratios of $(360° - \theta)$ in terms of θ, for all the values of θ**
   $\sin(360° - \theta) = -\sin\theta$
   $\cos(360° - \theta) = +\cos\theta$
   $\tan(360° - \theta) = -\tan\theta$
   $\cot(360° - \theta) = -\cot\theta$
   $\sec(360° - \theta) = +\sec\theta$
   $\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$
   Note: T-ratios of $(360° - \theta)$ and those of $(-\theta)$ are the same.
   Examples:
   $\sin 330^{\circ} = \sin(360° - 30°) = -\sin 30° = -\frac{1}{2}$
   $\sec 300^{\circ} = \sec(360° - 60°) = \sec 60° = 2$
   $co\sec 315^{\circ} = \operatorname{cosec}(360° - 45°) = -cosec45° = -\sqrt{2}$

9. **T-ratios of $(360° + \theta)$ in terms of θ, for all the values of θ**
   $\sin(360° + \theta) = +\sin\theta$
   $\cos(360° + \theta) = +\cos\theta$
   $\tan(360° + \theta) = +\tan\theta$
   $\cot(360° + \theta) = +\cot\theta$
   $\sec(360° + \theta) = +\sec\theta$
   $\operatorname{cosec}(360° + \theta) = +\operatorname{cosec}\theta$

Note:

1. T-ratios of $(360° + \theta)$ or $(2\pi + \theta)$ and those of θ are same.
2. T-ratios of $(n \times 360° + \theta)$, where $n = 1, 2, 3, \ldots$ also will be the same as that of θ.
3. In general, $\sin(n\pi + \theta) = (-1)^n sin\theta$
   $\cos(n\pi + \theta) = (-1)^n cos\theta$
   $\tan(n\pi + \theta) = tan\theta$
4. Similarly, $\sin\left(n\frac{\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}} cos\theta$
   $\cos\left(n\frac{\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}} sin\theta$
   $\tan\left(n\frac{\pi}{2} + \theta\right) = -cot\theta$, where n is any odd integer

Some Solved Problems

Q-1 : State $cos302^{\circ}$ is positive or negative.

Sol:

$\cos(3\times 90^{\circ}+32^{\circ})=sin32°$

$\therefore \sin 32^{\circ}$ lies in 1st quadrant, and by ASTC rule, all T-ratios are positive in 1st quadrant.

So, $\cos 302°$ is positive sign.

Q-2 : Find the value of $sin1230^{\circ}$.

Sol:

$sin1230^{\circ}=\left(\sin 3\times 360^{\circ}+150^{\circ}\right)=sin150^{\circ}$

$=\sin(180^{\circ}-30^{\circ})=\sin\ 30°=1/2$

Q-3: Express $sin\ 1185^{\circ}$ as the trigonometric ratio of some acute angle**.**

Sol:

$\sin 1185°=sin(13\ \times\ 90°\ \ +15°)$

$=(-1)^{\frac{13-1}{2}}\cos 15°=(-1)^{6}\,cos15°=cos15°$

Q-4 : Find the value of $\log tan17°+logtan37°+logtan53°+logtan73°$

Sol:

$\log tan17°+logtan37°+logtan53°+logtan73°$

$=\log tan17°\ tan37°\ tan53°\ tan73°$

$=logtan17°\ tan37°\ \tan(90°-37°)\ \tan(90°-17°)$

$=logtan17°tan37°cot37°cot17°$

$=log1=0 \qquad [\because\ tan17°cot17°=1\ and\ tan37°cot37°=1]$

Q-5: Show that $\frac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)}=-1.$

Proof:

LHS= $\frac{cos(90°+\theta)\,sec(-\theta)\,tan(180°-\theta)}{sec(360°-\theta)\,sin(180°+\theta)\,cot(90°-\theta)}=\frac{-sin\theta\ .\ sec\theta\ \ (-tan\theta)}{sec\theta\ \ (-sin\theta)\ \ tan\theta}=-1$

Even Function

A function $f(x)$ is said to be an even function of x, if $f(-x)=f(x)$.

Examples:

1. Let $f(x)=\cos x$, then $f(-x)=\cos(-x)=\cos x$

$\therefore\ f(x)=f(-x)=\cos x$.

Hence, $f(x)=\cos x$ is an even function.

2. Let $f(x)=\sin x\tan x$,

then $f(-x)=sin(-x)\,tan(-x)=(-\sin x)(-\tan x)=\sin x\tan x$

$\therefore\ f(x)=f(-x)=sin\,x\,tan\,x$.

Hence, $f(x)=sin\,x\,tan\,x$ is an even function.

3. Let $f(x)=1+x^{4}+\cot^{2}x$,

then $f(-x)=1+(-x)^{4}+\{cot(-x)\}^{2}=1+x^{4}+(-\,cot\,x)^{2}$

$=1+x^{4}+cot^{2}\,x$

$\therefore\ f(x) = f(-x) = 1 + x^4 + cot^2 x$.
Hence, $f(x) = 1 + x^4 + cot^2 x$ is an even function.

4. Let $f(x) = cos2x$

Then, $f(-x) = \cos 2(-x) = \cos 2x = f(x)$ $\quad$ [As $\cos(-\theta) = \cos\theta$]
So, $f(x) = cos2x$ is an even function

Odd Function

A function $f(x)$ is said to be an odd function of x, if $f(-x) = -f(x)$.
Example:

1. Let $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x = -f(x)$

So, $f(x) = \sin x$ is an odd function.

2. Let $f(x) = \tan x$, then $f(-x) = \tan(-x) = -\tan x = -f(x)$

So, $f(x) = \tan x$ is an odd function.

3. Let $f(x) = x^3 + cosec\ x$,

Then, $f(-x) = (-x)^3 + cosec\left(-x\right) = (-x^3) + (-cosec\ x)$
$= -(x^3 + cosec\ x) = -f(x)$
Hence, $f(x) = x^3 + cosec\ x$ is an odd function.

4. But, $Let\ f(x) = sin3x + 5$

Then, $f(-x) = sin(-3x) + 5 = -sin\,3x + 5$
Here, $f(-x)$ expressed neither as $f(x)$ nor as $-f(x)$.
Hence, $f(x) = sin3x + 5$ is neither an odd function nor an even function.

Theorem-1: (Addition Theorems)

(i) $\boldsymbol{sin(A + B) = sinA\ cosB + cosA\ sinB}$

(ii)

(iii) $\boldsymbol{tan(A + B) = \frac{tanA + tanB}{1 - tanA\ tanB}}$

(iv) $\boldsymbol{\cot(A + B) = \frac{cotA\ cotB - 1}{cotB + cotA}}$

Proof : Let the revolving line OM starting from the line OX make an angle ∠XOM=A and then further move to make $\angle MON = B,$
So that $\angle XON = A + B$
Let "P" be any point on the line ON.
Draw $PR \perp OX$, $PT \perp OM$, $TQ \perp PR$ and $TS \perp OX$
Then $\angle QPT = 90° - \angle PTQ = \angle QTO = \angle XOM = A$
We have from Δ OPR

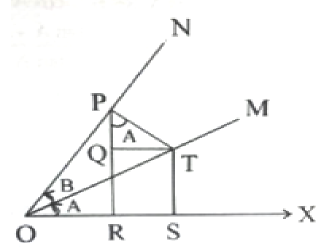


Fig 2.11

(i) $sin(A + B) = \frac{RP}{OP} = \frac{QR+PQ}{OP} = \frac{TS+PQ}{OP}$ $\quad$ $(\because QR = TS)$
$= \frac{TS}{OP} + \frac{PQ}{OP} = \left(\frac{TS}{OT}\right)\left(\frac{OT}{OP}\right) + \left(\frac{PQ}{PT}\right)\left(\frac{PT}{OP}\right)$
$= sinA\ cosB + cosA\ sinB$

(ii) $\cos(A + B) = \frac{QR}{OP} = \frac{OS-RS}{OP} = \frac{OS}{OP} - \frac{RS}{OP}$
$= \left(\frac{OS}{OT}\right)\left(\frac{OT}{OP}\right) - \left(\frac{QT}{PT}\right)\left(\frac{PT}{OP}\right)$
$= cosA\ cosB - sinA\ sinB$

(iii) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{sinA\ cosB + cosA\ sinB}{cosA\ cosB - sinA\ sinB}$

Dividing numerator and denominator by $\cos A \,.\, \cos B$

$$= \frac{\frac{sinAcosB}{cosAcosB} + \frac{cosAsinB}{cosAcosB}}{\frac{cosAcosB}{cosAcosB} - \frac{sinAsinB}{cosAcosB}} = \frac{tanA + tanB}{1 - tanA\ tanB}$$

Similarly, we can prove the followings theorems.

Theorem-2

(i) $\boldsymbol{\sin(A-B) = sinA\ cosB - cosA\ sinB}$

(ii) $\boldsymbol{\cos(A-B) = \cos A\ cosB + sinA\ sinB}$

(iii) $\boldsymbol{\tan(A-B) = \frac{tanA - tanB}{1 + tanA\ tanB}}$

The above theorems can be proved, by replacing B with –B in theorem-1.

Theorem-3

$\boldsymbol{\tan(A+B+C) = \frac{tanA + tanB + tanC - tanA\ tanB\ tanC}{1 - tanA\ tanB - tanB\ tanC - tanC\ tanA}}$ **, for** $\boldsymbol{A, B, C \epsilon R}$

Theorem-4

(i) $\boldsymbol{sin(A+B)\, sin(A-B) = sin^2A - sin^2B = cos^2B - cos^2A}$

(ii) $\boldsymbol{cos(A+B)\, cos(A-B) = cos^2A - sin^2B = cos^2B - sin^2A}$

Proof :

(i) $sin(A+B)\, sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(sin\, A\, cos\, B - cos\, A\, sin\, B)$

$= sin^2 A\, cos^2 B - cos^2 A\, sin^2 B$

$= sin^2 A\,(1 - sin^2 B) - (1 - sin^2 A)\, sin^2 B$

$= sin^2 A - sin^2 A\, sin^2 B - sin^2 B + sin^2 A\, sin^2 B$

$= sin^2 A - sin^2 B$ (Proved)

$= (1 - cos^2 A) - (1 - cos^2 B)$

$= cos^2 B - cos^2 A$ (Proved)

(ii) $cos(A+B)\, cos(A-B)$

$= (\cos A \cos B - \sin A \sin B)(cos\, A\, cos\, B + sin\, A\, sin\, B)$

$= cos^2 A\, cos^2 B - sin^2 A\, sin^2 B$

$= cos^2 A\,(1 - sin^2 B) - (1 - cos^2 A)\, sin^2 B$

$= cos^2 A - cos^2 A\, sin^2 B - sin^2 B + cos^2 A\, sin^2 B$

$= cos^2 A - sin^2 B$ (Proved)

$= (1 - sin^2 A) - (1 - cos^2 B)$

$= cos^2 B - sin^2 A$ (Proved)

Some Solved Problems

Q.1: Find the value of $\cos 15°$ **.**

Sol:

$cos15° = \cos(45° - 30°) = cos45°\ \ cos30° + sin45°\ \ sin30°$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Q-2: Find the value of $cos50°\ cos40° - sin50°\ sin40°$.

Sol:

$cos50°\ cos40° - sin50°\ sin40° = \cos(50° + 40°) = cos90° = 0$ [Use $\cos(A + B)$]

Q-3: If $\tan A = \frac{1}{2}$ and $tanB = \frac{1}{3}$, find the value of $\tan(A + B)$**.**

Sol:

We know that, $\tan(A + B) = \frac{tanA+tanB}{1-tanAtanB} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{5/6}{5/6} = 1.$

Q-4: Prove that $\frac{cos9° + sin9°}{cos9° - sin9°} = tan54°$

Proof:

LHS $= \frac{cos9°+sin9°}{cos9°-sin9°} = \frac{\frac{cos9°}{cos9°}+\frac{sin9°}{cos9°}}{\frac{cos9°}{cos9°}-\frac{sin9°}{cos9°}}$ (Dividing throughout by cos9°)

$$= \frac{1+tan9°}{1-tan9°} = \frac{tan45°+tan9°}{1-tan45°tan9°} \quad [\because tan45° = 1]$$

$$= ta\,n(45° + 9°) = tan54°$$ =RHS

Q-5: Prove that $sinA\ \sin(B - C) + sinB\ sin(C - A) + sinC\ sin(A - B) = 0$

Proof:

L.H.S: $sinA \sin(B - C) + sinB\ sin(C - A) + sinC\ sin(A - B)$

$= sinA(sinB\ cosC - cosB\ sinC) + sinB(sinC\ cosA - cosC\ sinA)$

$+sinC(sinA\ cosB - cosAsinB)$

$= sinA\ sinB\ cosC - sinA\ cosB\ sinC + sinB\ sinC\ cosA - sinBcosC\ sinA$

$+sinCsin - sinC\ cosA\ sinB$

(All the terms are cancelled with each other)

$= 0$ =R.H.S

Transformation of a Product into a Sum or Difference, and Vice-versa

(i) $\mathbf{\sin(A + B) + \sin(A - B) = 2\sin A\cos B}$

(ii) $\mathbf{\sin(A + B) - \sin(A - B) = 2\cos A\sin B}$

(iii) $\mathbf{\cos(A + B) + \cos(A - B) = 2\cos A\cos B}$

(iv) $\mathbf{\cos(A + B) - \cos(A - B) = -2\sin A\sin B}$

Proof:

From above established theorems 1 and 2,

(i) $sin(A+B) = sinA\ cosB + cosA\ sinB$
(ii) $\sin(A-B) = sinA\ cosB - cosA\ sinB$
(iii) $\cos(A+B) = \text{cosA cosB} - \text{sinA sinB}$
(iv) $\cos(A-B) = \cos A\ cosB + sinA\ sinB$

Adding (i) and (ii),

$\sin(A+B) + \sin(A-B) = (sinA\ cosB + cosA\ sinB) + (sinA\ cosB - cosA\ sinB)$
$= 2\sin A \cos B$

Subtracting (ii) from (i),

$\sin(A+B) - \sin(A-B) = (sinA\ cosB + cosA\ sinB) - (sinA\ cosB - cosA\ sinB)$
$= 2\cos A \sin B$

Again, Adding (iii) and (iv),

$\cos(A+B) + \cos(A-B) = (\text{cosA cosB} - \text{sinA sinB}) + (\cos A\ cosB + sinA\ sinB)$
$= 2\cos A \cos B$

Subtracting (iv) from (iii),

$\cos(A+B) - \cos(A-B) = (\text{cosA cosB} - \text{sinA sinB}) - (\cos A\ cosB + sinA\ sinB)$
$= -2\sin A \sin B$ (Proved)

Note :

Let $A+B=C$ and $A-B=D$

Then, $2A = C+D$ or $A = \frac{C+D}{2}$

and $2B = C-D$ or $B = \frac{C-D}{2}$

Putting the above values of A and B, in above four formulae, we get

$$\mathbf{\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)}$$
$$\mathbf{\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)}$$

Some Solved Problems

Q-1 : Prove that $sin50° - sin70° + sin10° = 0$.

Proof:

L.H.S= $sin50° - sin70° + sin10°$
$= \sin(60° - 10°) - \sin(60° + 10°) + sin10°$
$= -2cos60°\ sin10° + sin10°$ $(\because \sin(A-B) - \sin(A+B) = 2cosA\ sinB)$
$= -2 \times \frac{1}{2}\ sin10° + sin10°$
$= -sin10° + sin10° = 0$ =RHS

Q-2: Prove $\sin 10^{\circ} + \sin 20^{\circ} + \sin 40^{\circ} + \sin 50^{\circ} = \sin 70^{\circ} + \sin 80^{\circ}$

Proof:

LHS $= \sin 10^{\circ} + \sin 20^{\circ} + \sin 40^{\circ} + \sin 50^{\circ} = \sin 70^{\circ} + \sin 80^{\circ}$
$= (\sin 10^{\circ} + \sin 50^{\circ}) + (\sin 20^{\circ} + \sin 40^{\circ})$

$= 2\sin\left(\frac{10^\circ + 50^\circ}{2}\right)\cos\left(\frac{10^\circ - 50^\circ}{2}\right) + 2\sin\left(\frac{20^\circ + 40^\circ}{2}\right)\cos\left(\frac{20^\circ - 40^\circ}{2}\right)$
$= 2\sin 30^\circ \cos 20^\circ + 2\sin 30^\circ \cos 10^\circ$
$= 2\sin 30^\circ (\cos 20^\circ + \cos 10^\circ)$
$= 2\ \frac{1}{2}(\cos 20^\circ + \cos 10^\circ)$
$= \cos 20^\circ + \cos 10^\circ$
$= \cos(90^\circ - 70^\circ) + \cos(90^\circ - 80^\circ)$
$= \cos 70^\circ + \cos 80^\circ$ =RHS

Q-3: Prove that $\frac{cos7\alpha + cos3\alpha - cos5\alpha - cos\alpha}{sin7\alpha - sin3\alpha - sin5\alpha + sin\alpha} = cot2\alpha$

Proof:

LHS.: $= \frac{cos7\alpha + cos3\alpha - cos5\alpha - cos\alpha}{sin7\alpha - sin3\alpha - sin5\alpha + sin\alpha}$

$= \frac{(cos7\alpha + cos3\alpha) - (cos5\alpha + cos\alpha)}{(sin7\alpha - sin3\alpha) - (sin5\alpha - sin\alpha)}$

$= \frac{2\cos\left(\frac{7\alpha+3\alpha}{2}\right)\cos\left(\frac{7\alpha-3\alpha}{2}\right) - 2\cos\left(\frac{5\alpha+\alpha}{2}\right)\cos\left(\frac{5\alpha-\alpha}{2}\right)}{2\cos\left(\frac{7\alpha+3\alpha}{2}\right)\sin\left(\frac{7\alpha-3\alpha}{2}\right) - 2\cos\left(\frac{5\alpha+\alpha}{2}\right)\sin\left(\frac{5\alpha-\alpha}{2}\right)}$

$= \frac{2cos4\alpha\ cos2\alpha - 2cos3\alpha\ cos2\alpha}{2cos4\alpha\ sin2\alpha - 2cos3\alpha\ sin2\alpha}$

$= \frac{2cos2\alpha(cos4\alpha - cos3\alpha)}{2sin2\alpha(cos4\alpha - cos3\alpha)}$

$= \frac{cos2\alpha}{sin2\alpha} = cot2\alpha$ = RHS

Q-4: If $sinA = KsinB$, Prove that $tan\frac{1}{2}(A - B) = \frac{K-1}{K+1}\ tan\frac{1}{2}(A + B)$.

Proof:

Given $sinA = KsinB$

$\Rightarrow \frac{sinA}{sinB} = \frac{K}{1}$

$\Rightarrow \frac{sinA - sinB}{sinA + sinB} = \frac{K-1}{K+1}$

$\Rightarrow \frac{2\ cos\frac{A+B}{2}\ sin\frac{A-B}{2}}{2\ sin\frac{A+B}{2}\ cos\frac{A-B}{2}} = \frac{K-1}{K+1}$

$\Rightarrow cot\frac{A+B}{2}\tan\frac{A-B}{2} = \frac{K-1}{K+1}$

$\Rightarrow tan\frac{1}{2}(A - B) = \frac{K-1}{K+1}\ tan\frac{1}{2}(A + B)$

Q-5: If $A + B + C = \pi$, Prove that $sin2A + sin2B + sin2C = 4sinAsinBsinC$.

Proof:

L.H.S.= $sin2A + sin2B + sin2C$
$= (sin2A + sin2B) + sin2C$
$= 2\sin(A + B)\cos(A - B) + 2sinC\ cosC$

$= 2sinC\ cos(A - B) + 2sinC\ cosC$ $[\because A + B = \pi - C, \sin(A + B) = \sin(\pi - C) = sinC]$
$= 2sinC\ \{\cos(A - B) + cosC\}$
$= 2sinC(\cos(A - B) - \cos(A + B))$ $[\because cosC = \cos\{\pi - (A + B)\} = -\cos(A + B)]$
$= 2sinC\ (2sinA\ sinB)$
$= 4sinA\ sinB\ sinC =$ RHS

Compound, Multiple and Sub Multiple Angles

Multiple and Sub Multiple Arguments : For an argument (variable) θ usually $2\theta, 3\theta$ etc. are called its multiples and $\theta/2$, $\theta/3$ etc. are called its sub multiples. For arguments θ and $\emptyset$, $\theta + \emptyset$, $\theta - \emptyset$ are called the compound arguments.

Theorem-1

(i) $\boldsymbol{sin2A = 2sinA\ cosA}$

(ii) $\boldsymbol{cos2A = cos^2A - sin^2A = 2cos^2A - 1 = 1 - 2sin^2 A}$

(iii) $\boldsymbol{tan2A = \frac{2tanA}{1-tan^2A};\ A \neq (2n+1)\frac{\pi}{2}}$

Proof:

(i) According to addition theorem, $\sin(A + B) = sinA\ cosB + cosA\ sinB$

Replace the angle B by A, $\sin(A + A) = sinA\ cosA + cosA\ sinA$

Or, $\mathrm{sin2A = 2sinA\ cosA}$

(ii) According to addition theorem, $\cos(A + B) = \mathrm{cosA\ cosB - sinA\ sinB}$

Replace the angle B by A, $\cos(A + A) = \mathrm{cosA\ cosA - sinA\ sinA}$

Or, $\cos 2A = cos^2A - sin^2A$

Again, by using the identity, $sin^2A + cos^2A = 1$ in above formula,

Or, $\cos 2A = cos^2A - (1 - cos^2A) = 2cos^2A - 1$

and, $\cos 2A = (1 - sin^2A) - sin^2A = 1 - 2sin^2A$

Again, by using the identity, $sin^2A + cos^2A = 1$

(iii) According to addition theorem, $tan(A + B) = \frac{tanA + tanB}{1 - tanA\ tanB}$

Replace the angle B by A, $tan(A + A) = \frac{tanA + tanA}{1 - tanA\ tanA}$

Or, $tan\ 2A = \frac{2\tan A}{1 - tan^2A}$

Theorem-2

(i) $\boldsymbol{sin3A = 3sinA - 4sin^3A}$

(ii) $\boldsymbol{\cos 3A = 4cos^3A - 3cosA}$

(iii)

Proof:

(i) $\sin 3A = \sin(2A + A) = sin2A\ cosA + cos2A\ sinA$
$= (2sinA\ cosA\ )cosA + (1 - 2sin^2A)sinA$
$= 2sinA\ cos^2A + sinA - 2sin^3A$
$= 2sinA(1 - sin^2A) + sinA - 2sin^3A$
$= 3sinA - 4sin^3A$

(ii) $\cos 3A = \cos(3A + A) = \cos 2A \cos A - \sin 2A \sin A$

$= (2cos^2 A - 1)\ \cos A - (2\sin A\cos A)\sin A$
$= 2cos^3 A - \cos A - 2\ sin^2 A\cos A$
$= 2cos^3 A - \cos A - 2\ (1 - cos^2 A)\cos A$
$= 2cos^3 A - \cos A - 2\ \cos A + 2\ cos^3 A$
$= 4\ cos^3 A - 3\cos A$

(iii) $\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

$$= \frac{\left(\frac{2\tan A}{1 - tan^2 A}\right) + \tan A}{1 - \left(\frac{2\tan A}{1 - tan^2 A}\right)\tan A}$$

$$= \frac{3\tan A - tan^3 A}{1 - 3\ tan^2 A}$$

Note : Replace A by $A/2$, in Theorem-1, the followings can be proved.

(i) $\boldsymbol{sinA = 2sin\frac{A}{2}\ cos\frac{A}{2}}$

(ii) $\boldsymbol{cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2} = 2cos^2\frac{A}{2} - 1 = 1 - 2sin^2\frac{A}{2}}$

(iii) $\boldsymbol{tanA = \frac{2tan\frac{A}{2}}{1 - tan^2\frac{A}{2}}}$

Note : Replace A by $A/3$, in Theorem-1, the followings can be derived.

(iv) $\boldsymbol{sin\theta = 3sin\frac{\theta}{3} - 4sin^3\frac{\theta}{3}}$

(v) $\boldsymbol{cos\theta = 4cos^3\frac{\theta}{3} - 3cos\frac{\theta}{3}}$

(vi) $\boldsymbol{tan\theta = \frac{3tan\frac{\theta}{3} - tan^3\frac{\theta}{3}}{1 - 3tan^2\frac{\theta}{3}}}$

Some Solved Problems

Q-1: Prove that $\frac{cotA - tanA}{cotA + tanA} = cos2A$.

Proof:

LHS: $\frac{cotA - tanA}{cotA + tanA} = \frac{\frac{cosA}{sinA} - \frac{sinA}{cosA}}{\frac{cosA}{sinA} + \frac{sinA}{cosA}} = \frac{\frac{cos^2A - sin^2A}{sinA\ cosA}}{\frac{cos^2A + sin^2A}{sinA\ cosA}}$

$= \frac{cos^2A - sin^2A}{sinA\ cosA} \times \frac{sinA\ cosA}{cos^2A + sin^2A} = cos2A$ =RHS

Q-2: Prove that $cot\frac{A}{2} = \frac{sinA}{1 - cosA}$

Proof:

R.H.S= $\frac{sinA}{1 - cosA} = \frac{2sin\frac{A}{2}cos\frac{A}{2}}{2sin^2\frac{A}{2}} = \frac{cos\frac{A}{2}}{sin\frac{A}{2}} = cot\frac{A}{2}$ =LHS

Q- 3: Prove that $cotA - cosec2A = cot2A$

Proof:

L.H.S= $cotA - cosec2A = \frac{cosA}{sinA} - \frac{1}{sin2A}$

$$= \frac{cosA}{sinA} - \frac{1}{2sinA\ cosA} = \frac{2cos^2A - 1}{2sinA\ cosA}$$

$$= \frac{cos2A}{sin2A} = cot2A$$

Q- 4: Find the value of $sin20°(3 - 4cos^2 70°)$?

Sol:

$$sin20°(3 - 4cos^2 70°) = sin20°[3 - 4cos^2(90° - 20°)]$$

$$= sin20°(3 - 4sin^2 20°)$$

$$= 3sin20° - 4sin^3 20°$$

$$= \sin(3 \times 20°) = sin60° = \frac{\sqrt{3}}{2}$$

Q- 5: Prove that $cos^6A - sin^6A = cos2A\left(1 - \frac{1}{4}sin^2 2A\right)$

Proof:

LHS $= cos^6A - sin^6A = (cos^2A)^3 - (sin^2A)^3$

$$= (cos^2A - sin^2A)(cos^4A + cos^2A\ sin^2A + sin^4A)$$

$$= cos2A\{(sin^2A)^2 + (cos^2A)^2 + sin^2A\ cos^2A\}$$

$$= cos2A\left\{\begin{matrix}(sin^2A)^2 + (cos^2A)^2 + 2sin^2A\ cos^2A \\ -2sin^2A\ cos^2A\ + sin^2A\ cos^2A\end{matrix}\right\}$$

$$= cos2A\{(cos^2A + sin^2A)^2 - sin^2Acos^2A\}$$

$$= cos2A\{1 - (\sin A \cos A)^2\}$$

$$= cos2A\left\{1 - \left(\frac{1}{2}.2\sin A\cos A\right)^2\right\}$$

$= \cos 2A\left(1 - \frac{1}{4}sin^2 2A\right)$ =RHS

Q- 6: Prove that $sin50° - sin70° + sin10° = 0$

Proof:

LHS $= sin50° - sin70° + sin10°$

$$= \sin(60° - 10°) - \sin(60° + 10°) + sin10°$$

$$= -2cos60°\ sin10° + sin10°$$

$$= -2 \times \frac{1}{2}\ sin10° + sin10°$$

$= -sin10° + sin10° = 0 =$ RHS

Q- 7: Find the value of $2sin67\frac{1}{2}^{\circ}\ cos22\frac{1}{2}^{\circ}$.

Ans:

$2\sin\left(90° - 22\frac{1}{2}^{\circ}\right)\cos 22\frac{1}{2}^{\circ} = 2cos22\frac{1}{2}^{\circ}\ cos22\frac{1}{2}^{\circ}$ $[\because \sin(90° - \theta) = cos\theta]$

$$= 2cos^2\ 22\frac{1}{2}^{\circ}$$

$$= 2cos^2 22\frac{1}{2}^{\circ} - 1 + 1$$

$= cos2 \times 22\frac{1}{2}{}^\circ + 1$

$= cos45^\circ + 1 = \frac{1}{\sqrt{2}} + 1$

Q- 8: Prove that $cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

Proof:

we know that $1 + cos\theta = 2cos^2\frac{\theta}{2} \ldots\ldots\ldots\ldots..(1)$

Put $\theta = \frac{\pi}{4}$, $1 + cos\frac{\pi}{4} = 2cos^2\frac{\pi}{8}$

$\Rightarrow 1 + \frac{1}{\sqrt{2}} = 2cos^2\frac{\pi}{8}$

$\Rightarrow 4cos^2\frac{\pi}{8} = 2\left(1+\frac{1}{\sqrt{2}}\right) = 2+\sqrt{2}$

$\Rightarrow 2cos\frac{\pi}{8} = \sqrt{2+\sqrt{2}}$

put $\theta = \frac{\pi}{8}$ in eqn.(1), we get

$2cos^2\frac{\pi}{16} = 1 + cos\frac{\pi}{8}$

$\Rightarrow 4cos^2\frac{\pi}{16} = 2 + 2\,cos\frac{\pi}{8}$

$\Rightarrow \left(2cos\frac{\pi}{16}\right)^2 = 2+\sqrt{2+\sqrt{2}}$

$\Rightarrow 2cos\frac{\pi}{16} = \sqrt{2+\sqrt{2+\sqrt{2}}}$ (Proved)